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EXACT SOLUTIONS OF THE AXISYMMETRIC EQUATIONS OF MOTION OF A VISCOUS HEAT-CONDUCTING PERFECT GAS DESCRIBED BY SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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All invariant solutions of rank 1 of the two-dimensional equations of motion of a heat-conducting perfect gas with a polytropic equation of state are described. A sufficient condition for reducibility of regular, partially invariant solutions of rank 1 and defect 1 to invariant solutions is given.

1. Formulation of the Problem. We consider the following equations of axisymmetric motion of a heat-conducting perfect gas with a polytropic equation of state:

$$\rho(u_t + uu_r + wu_z) = -p_r + \frac{2}{3}\Big(\mu\Big(2u_r - w_z - \frac{u}{r}\Big)\Big)_r + (\mu(u_z + w_r))_z + 2\mu\Big(\frac{u}{r}\Big)_r; \tag{1.1}$$

$$\rho(w_t + uw_r + ww_z) = -p_z + (\mu(u_z + w_r))_r + \frac{2}{3}\Big(\mu\Big(2w_z - u_r - \frac{u}{r}\Big)\Big)_z + \frac{\mu}{r}(u_z + w_r); \tag{1.2}$$

$$\rho_t + (u\rho)_r + (w\rho)_z + \frac{u\rho}{r} = 0; \tag{1.3}$$

$$p_t + up_r + wp_z + \gamma p\Big(u_r + w_z + \frac{u}{r}\Big) = \frac{\gamma - 1}{R}k_0\Big(\Big(\mu\Big(\frac{p}{\rho}\Big)_r\Big)_r + \Big(\mu\Big(\frac{p}{\rho}\Big)_z\Big)_z + \frac{\mu}{r}\Big(\frac{p}{\rho}\Big)_r\Big)$$
$$+(\gamma - 1)\mu\Big(\frac{4}{3}\Big(u_r^2 + w_z^2 - u_r w_z + \frac{u}{r}\Big(\frac{u}{r} - u_r - w_z\Big)\Big) + (w_r + u_z)^2\Big). \tag{1.4}$$

Here ρ is the density, p is the pressure, $\mu = (p/\rho)^\omega$ is the viscosity, $k_0\mu$ is the thermal conductivity, γ is the adiabatic exponent, and R is the gas constant.

The aim of the present paper is to derive invariant solutions of rank 1 of system (1.1)–(1.4) [1] and to obtain a sufficient condition for reducibility of regular, partially invariant solutions of rank 1 and defect 1 to invariant solutions [2]. All such solutions are described by systems of ordinary differential equations.

In [3], it is shown that Eqs. (1.1)–(1.4) admit the Lie algebra L_5 with the basis

$$X_1 = \partial_z, \qquad X_2 = t\partial_z + \partial_w, \qquad X_3 = \partial_t, \qquad X_4 = t\partial_t + r\partial_r + z\partial_z - \rho\partial_\rho - p\partial_p,$$
$$X_5 = r\partial_r + z\partial_z + u\partial_u + w\partial_w + 2(\omega - 1)\rho\partial_\rho + 2\omega p\partial_p. \tag{1.5}$$

In the same paper, the normalized optimal system of subalgebras of the Lie algebra L_5 that is used in the present paper is constructed. The Lie group of transformations associated with this algebra is denoted by G_5.

2. Invariant Solutions of Rank 1. It is known that invariant solutions of rank 1 of system (1.1)–(1.4) are constructed from two-dimensional subalgebras satisfying the necessary condition for the existence of an invariant solution. Table 1 gives all these subalgebras from the optimal system. The basis of the algebra H is denoted by the corresponding operator numbers in (1.5). For example, the basis of the algebra $\{X_3, X_4 + \alpha X_5\}$ is indicated as $\{3,\ 4 + \alpha 5\}$.

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TABLE 1

No.	H	Solution
1	$3,\ 4+\alpha 5$ ($\alpha \neq -1$)	$u = r^{\alpha/(\alpha+1)}u_1(\xi)$, $w = r^{\alpha/(\alpha+1)}w_1(\xi)$, $\rho = r^{(2\alpha(\omega-1)-1)/(\alpha+1)}\rho_1(\xi)$, $p = r^{(2\alpha\omega-1)/(\alpha+1)}p_1(\xi)$, $\xi = z/r$
2	$2,\ 4+\alpha 5$ ($\alpha \neq -1$)	$u = t^{\alpha}u_1(\xi)$, $w = z/t + t^{\alpha}w_1(\xi)$, $\rho = t^{2\alpha(\omega-1)-1}\rho_1(\xi)$, $p = t^{2\alpha\omega-1}p_1(\xi)$, $\xi = rt^{-\alpha-1}$
3	$1,\ 4+\alpha 5$	$u = t^{\alpha}u_1(\xi)$, $w = t^{\alpha}w_1(\xi)$, $\rho = t^{2\alpha(\omega-1)-1}\rho_1(\xi)$, $p = t^{2\alpha\omega-1}p_1(\xi)$, $\xi = rt^{-\alpha-1}$
4	$4,\ 5$	$u = rt^{-1}u_1(\xi)$, $w = rt^{-1}w_1(\xi)$, $\rho = r^{2(\omega-1)}t^{1-2\omega}\rho_1(\xi)$, $p = r^{2\omega}t^{-1-2\omega}p_1(\xi)$, $\xi = z/r$
5	$3,\ 1+4-5$	$u = \exp(-z)u_1(r)$, $w = \exp(-z)w_1(r)$, $\rho = \exp((1-2\omega)z)\rho_1(r)$, $p = \exp(-(1+2\omega)z)p_1(r)$
6	$2,\ 1+4-5$	$u = u_1(r)/t$, $w = (z - \ln t + w_1(r))/t$, $\rho = t^{1-2\omega}\rho_1(r)$, $p = t^{-1-2\omega}p_1(r)$
7	$2+3,\ 4+5$	$u = r^{1/2}u_1(\xi)$, $w = t + r^{1/2}w_1(\xi)$, $\rho = r^{\omega-3/2}\rho_1(\xi)$, $p = r^{\omega-1/2}p_1(\xi)$, $\xi = (t^2-2z)/r$
8	$1,\ 2+4$	$u = u(\xi)$, $w = \ln t + w_1(\xi)$, $\rho = \rho_1(\xi)/t$, $p = p_1(\xi)/t$, $\xi = r/t$
9	$1,\ 3+5$	$u = \exp(t)u_1(\xi)$, $w = \exp(t)w_1(\xi)$, $\rho = \exp(2(\omega-1)t)\rho_1(\xi)$, $p = \exp(2\omega t)p_1(\xi)$, $\xi = r\exp(-t)$
10	$1,\ 3-5$	$u = \exp(-t)u_1(\xi)$, $w = \exp(-t)w_1(\xi)$, $\rho = \exp(2(1-\omega)t)\rho_1(\xi)$, $p = \exp(-2\omega t)p_1(\xi)$, $\xi = r\exp(t)$
11	$3,\ 5$	$u = ru_1(\xi)$, $w = rw_1(\xi)$, $\rho = r^{2(\omega-1)}\rho_1(\xi)$, $p = r^{2\omega}p_1(\xi)$, $\xi = z/r$
12	$2,\ 5$	$u = ru_1(t)$, $w = (z + rw_1(t))/t$, $\rho = r^{2(\omega-1)}\rho_1(t)$, $p = r^{2\omega}p_1(t)$
13	$1,\ 5$	$u = ru_1(t)$, $w = rw_1(t)$, $\rho = r^{2(\omega-1)}\rho_1(t)$, $p = r^{2\omega}p_1(t)$
14	$1,\ 2+3$	$u = u(r)$, $w = t + w_1(r)$, $\rho = \rho(r)$, $p = p(r)$
15	$1,\ 3$	$u = u(r)$, $w = w(r)$, $\rho = \rho(r)$, $p = p(r)$

A particular solution of rank 1 can be obtained as follows. The general form of the solution for a particular subalgebra is substituted into system (1.1)–(1.4). This results in a quotient system which is a system of second-order (in some cases, first-order) differential equations. Quotient systems are not given here to save space.

The solutions constructed from subalgebras 1, 5, 11, and 15 describe steady gas flows, and the remaining solutions describe unsteady flows.

Generally, only the analog of Eq. (1.3) in a quotient system can be integrated analytically. However, in some particular cases, quotient systems can be analyzed more thoroughly. For three subalgebras, we give the result of integration for constant viscosity and thermal conductivity (i.e., for $\omega = 0$). Integration is carried out with accuracy up to the normalizer of the corresponding subalgebra in L_5.

Subalgebra 12. Integration of the quotient system yields

$$u_1 = \frac{1}{t+t_0}, \qquad w_1 = w_0\,\frac{\exp(t^2/2)}{t+t_0}, \qquad \rho_1 = \frac{1}{t}.$$

TABLE 2

No.	H	Invariant subgroups
1	$1,2,4+\alpha 5$	$rt^{-\alpha-1},\ ut^{-\alpha},\ \rho t^{1-2\alpha(\omega-1)},\ pt^{1-2\alpha\omega}$
2	$1,2,3+5$	$re^{-t},\ ue^{-t},\ \rho e^{2(1-\omega)t},\ pe^{-2\omega t}$
3	$1,2,3-5$	$re^{t},\ ue^{t},\ \rho e^{2(\omega-1)t},\ pe^{2\omega t}$
4	$1,2,3$	$r,\ u,\ \rho,\ p$
5	$1,2,5$	$t,\ u/r,\ \rho r^{2(1-\omega)},\ pr^{-2\omega}$

The function $p_1(t)$ is obtained from the equation

$$p_1' + \Big(\frac{2\gamma}{t+t_0} + \frac{\gamma}{t} - 4\,\frac{\gamma-1}{R}\,k_0 t\Big)p_1 = \frac{\gamma-1}{t^2(t+t_0)^2}\Big(\frac{4}{3}\,(1-t^2+t_0^2) + w_0^2\exp(t^2)\Big).$$

Subalgebra 13. Integration of the quotient system yields

$$u_1 = \frac{1}{t}, \qquad w_1 = w_0\,\frac{\exp(t)}{t}, \qquad \rho_1 = 1.$$

The function $p_1(t)$ is obtained from the equation

$$p_1' + \Big(\frac{2\gamma}{t} - 4\,\frac{\gamma-1}{R}\,k_0\Big)p_1 = \frac{\gamma-1}{t^2}\Big(\frac{4}{3} + w_0^2\exp(2t)\Big).$$

Subalgebra 15. Integration of the quotient system yields two integrals

$$ru\rho = c_1, \qquad w = w_0 r^{c_1}.$$

The remaining equations

$$\frac{4}{3}\Big(u' + \frac{u}{r}\Big)' - c_1\,\frac{u'}{r} - p' = 0,$$

$$up' + \gamma p\Big(u' + \frac{u}{r}\Big) = \frac{k_0(\gamma-1)}{c_1 R}\Big((rup)'' + \frac{(rup)'}{r}\Big) + (\gamma-1)\Big(\frac{4}{3}\Big(u'^2 - \frac{uu'}{r} + \frac{u^2}{r^2}\Big) + w_0^2 r^{2c_1}\Big)$$

are used to obtain the functions $u(r)$ and $p(r)$.

3. Partially Invariant Solutions Reducible to Invariant Solutions. Invariant solutions are constructed more easily than partially invariant solutions. Therefore, using criteria for sifting out partially invariant solutions reducible to invariant solutions, we can concentrate on constructing irreducible solutions. Here we consider a sufficient condition for reducibility of regular, partially invariant solutions of rank 1 and defect 1 to invariant solutions. Since all such solutions are reduced to invariant solutions of rank 1, it is not necessary to study them separately: all these solutions were described above.

Theorem. *If the universal invariant of the subgroup $H \subset G_5$ can be written as*

$$J = (\xi(t,r),\ A(t,r)u,\ B(t,r)\rho,\ C(t,r)p), \tag{3.1}$$

where ξ, A, B, and C are some functions, then the corresponding regular, partially invariant H-solution of rank 1 *and defect* 1 *of system* (1.1)–(1.4) *is reducible to an invariant solution.*

Proof. It is known that the rank and defect of a partially invariant solution are invariant with respect to the similarity transformation of subgroups. Analysis of the invariants of all subgroups shows that for each subgroup condition (3.1) is also invariant with respect to the similarity transformation of the subgroups. Therefore, it suffices to prove the theorem only for the subgroups from the optimal system. All the subgroups satisfying the condition (3.1) are listed in Table 2. The notation for the subalgebras is the same as in Table 1.

The general ideas of the proof are as follows. For each subgroup in Table 2, the invariants specify the general form of the solution. For $\partial\xi/\partial r \neq 0$, it follows from Eq. (3.1) that

$$w(t,r,z) = \varphi(\xi)\,\frac{z}{t} + v(t,\xi), \tag{3.2}$$

or

$$w(t,r,z) = \varphi(\xi)z + v(t,\xi). \tag{3.3}$$

Then, Eq.(1.4) takes the form

$$F(\xi) + \Big(f_1(t)\varphi' z + f_2(t)\,\frac{\partial v}{\partial \xi}\Big)^2 = 0, \tag{3.4}$$

where F, f_1, and f_2 are known functions (particular for each subgroup). It follows from (3.4) that $\varphi' = 0$.

For $\xi = t$, (3.2) or (3.3) takes the form $w(t,r,z) = \varphi(t)z + v(t,r)$, and (3.4) becomes

$$F(t) + \Big(\frac{\partial v}{\partial r}\Big)^2 = 0. \tag{3.5}$$

Integrals of Eqs. (3.4) and (3.5) are used to analyze Eq. (1.1). As a result, we obtain the general form of w, which is then used to determine the subgroup with respect to which the solution obtained is invariant.

Proofs for each particular subgroup are not given here. We note that all solutions are reduced to solutions that are invariant with respect to groups similar to subgroups 3, 6, 9, 10, 12, 13, and 15 in Table 1. The theorem is proved.

Among all subgroups from which it is possible to construct regular, partially invariant solutions of rank 1 and defect 1, only two subgroups do not satisfy condition (3.1). They yield partially invariant solutions that are not reducible to invariant solutions and are not considered here.

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